Generalized Laguerre polynomials and quantum mechanics

2000 J. Phys. A: Math. Gen. 33 7659

(http://iopscience.iop.org/0305-4470/33/42/401)

J. Phys. A: Math. Gen. **33** (2000) 7659–7660. Printed in the UK PII: S0305-4470(00)12764-7

COMMENT

Generalized Laguerre polynomials and quantum mechanics

Paul C Abbott

Department of Physics, University of Western Australia, Nedlands 6907, Australia

E-mail: `paul@physics.uwa.edu.au`

Received 22 March 2000

Abstract. We comment on serious errors in a recently published paper (El-Sayed A M A 1999 *J. Phys. A: Math. Gen.* **32** 8647–54).

Generalized Laguerre polynomials

The so-called 'generalized Laguerre polynomials' discussed in 'On the generalized Laguerre polynomials of arbitrary (fractional) orders and quantum mechanics' (El-Sayed 1999) are not polynomials. Even the name gives rise to confusion, since this terminology is already used. Moreover they are not, as claimed in the paper, square-integrable. The differential equation

$$x\,u''(x) + (1+\beta-x)\;u'(x) + \alpha\,u(x) = 0$$

is, of course, just a special case of Kummer's equation and has the confluent hypergeometric functions as solutions (Abramowitz and Stegun 1970, chapter 13). The particular solution

$$Ł_{\alpha}^{\beta}(x) = \frac{\Gamma\,(\alpha+\beta+1)}{\Gamma\,(\alpha+1)\,\Gamma\,(\beta+1)} M(-\alpha, \beta+1, x)$$

reduces to the standard generalized Laguerre polynomials (Abramowitz and Stegun 1970, section 13.6.9) when $\alpha = n = 0, 1, 2, \ldots$.

It is easy to show that the integral (equation (30) of El-Sayed 1999)

$$\int_0^\infty \mathrm{e}^{-x}\,x^{\beta}\,Ł_{\alpha}^{\beta}(x)^2\,\mathrm{d}x$$

diverges when α is not a non-negative integer. From the asymptotic expansion of $M(a, b, z)$ for $|z|$ large (a, b, fixed) (Abramowitz and Stegun 1970, section 13.5.1), $M(-\alpha, \beta+1, x)$ goes like

$$\frac{\Gamma(\beta+1)}{\Gamma(-\alpha)}\mathrm{e}^{x}x^{-\alpha-\beta-1}$$

leading to the following divergent integral

$$\int_0^\infty \mathrm{e}^{+x}\,x^{-2\alpha-\beta-2}\,\mathrm{d}x.$$

The obvious fallacy which led to the incorrect conclusion presented by El-Sayed (1999) is the argument leading to his equation (40). A bound for a function over a finite interval, [0, N], cannot be used to show that the function is square-integrable over $[0, \infty)$. As a simple counter-example, e^x is bounded by $M = e^N$ over [0, N] but

$$\int_0^\infty e^{-x}\, x^\beta\, (e^x)^2\, dx$$

is clearly divergent.

Quantum Mechanics

From the above comments it follows that the claims made in section 5 of El-Sayed (1999) are clearly wrong. His 'generalized Laguerre polynomials' do *not* 'enhance[s] the field of the definition of the solutions for the hydrogen atom', nor do they 'add a continuous spectrum in between the discrete spectrum' or 'open the question of the completeness of the solutions'. In fact, it was these rather surprising claims—which contradict the standard solutions presented in any undergraduate quantum physics text—that made me question the validity of the paper.

References

Abramowitz M and Stegun I 1970 *Handbook of Mathematical Functions* (New York: Dover)

El-Sayed A M A 1999 On the generalized Laguerre polynomials of arbitrary (fractional) orders and quantum mechanics *J. Phys. A: Math. Gen.* **32** 8647–54